Spin Waves

Introduction

The phenomenon of ferromagnetic resonance (FMR) is well presented in Kittel's « Introduction to Solid State Physics ». Consider a ferromagnetic thin film, or a single domain magnetic particles in the shape of an ellipsoid. Apply a uniform, static magnetic field H of the order of 1 to 20 kG. Set the sample in a microwave cavity. A resonance is observed at a frequency given by

$$\hbar\omega = g\mu_B \sqrt{[H + (N_x - N_z)M][H + (N_y - N_z)M]}$$

where it is implied that the z axis is along the applied field.
This equation can be derived from a semi-classical equation of motion for the magnetization $\vec{M}$:

$$\frac{d\vec{M}}{dt} = \gamma\vec{M} \times \vec{H}_{eff}$$

where the effective field takes into account the applied field and the dipolar field. This is a classical equation of motion for the angular momentum $\vec{J}$ that one relates to the magnetization by the quantum mechanical relation $\vec{M} = \gamma\vec{J}$.

The effective field can be derived from a magnetic energy:

$$E = -M_z HV + \frac{1}{2}\left(N_x M_x^2 + N_y M_y^2 + N_z M_z^2\right)$$

This approach can be generalized to a non-uniform magnetization. In this case an exchange energy must be taken into account, from which an exchange field is derived. It is possible to develop a continuum-type description of exchange. It is used to predict the magnetic configurations of magnetic structures, in particular of particles small enough that a single-domain behavior prevails. Aharoni has reviewed this field of research in his « Introduction to Ferromagnetism » (1997).

The equation of evolution for $\vec{M}$ as given above may appear surprising, since it involves the gyromagnetic ratio $\gamma = \frac{g\mu}{\hbar}$ of the electron, whereas $\vec{M}$ refers to a collective state of the electrons of the ferromagnet. However, this semi-classical approach was supported by a three-page paper of Luttinger and Kittel written in 1948. These authors started out from the classical expression for the energy above and inferred the hamiltonian :

$$H = -g\mu_B J_z H + \frac{1}{2}\frac{g^2\mu_B^{\,2}}{V}\left(N_x J_x^2 + N_y J_y^2 + N_z J_z^2\right)$$

by replacing the classical magnetic moment $\vec{M}V$ of the ellipsoid by

$$g\mu_B\vec{J} = \vec{M}V$$

That is, they represented the angular momentum of the correlated state of the ferromagnet by a giant angular moment of some 10^{15} spins. At normal temperatures, the eigenvalue m of Jz was estimated to be of about a factor 10^5 smaller than that of J. An equation of evolution was deduced for this limit of large J and m, with m much smaller than J. It was analogous to that of a harmonic oscillator. The spacing between the energy levels corresponded exactly to the classical energy given above. Morrish relies on this result to justify his classical treatment of ferromagnetic resonance, which constitutes a good, practical introduction to FMR. ("The physical principles of magnetism" Allan H. Morrish" New York [etc.] : Wiley, cop. 1965)

One-magnon eigenstates of a ferromagnet

(ref. Ashcroft, Mermin, Introduction to Solid State Physics)

We consider in the following a ferromagnet which can be described by an exchange hamiltonian of the form:

$$H = -\sum_{nearest} \sum_{neigbor} J\left(\vec{R}-\vec{R}'\right)\vec{S}\left(\vec{R}\right)\vec{S}\left(\vec{R}'\right) - Hg\mu_B \sum S_i^z$$

corresponding to a field H along the +z axis. This hamiltoninian would describe a ferromagnetic insulator with localized moments. Mattis has an extensive discussion of the exchange hamiltonian in chapter 2 of his "Quantum Theory of Magnetism ». If one considers spin waves of short wave length, one can expect this exchange interaction to dominate over other long range interactions such as the dipolar couplings among spins.

The ground state of this ferromagnet has all the spins pointing in the +z direction. We denote this ground state by $|0\rangle$. Now we consider states where the angular momentum at position **R** has its z component decreased by one unit, that is

$$\left|\vec{R}\right\rangle = \frac{1}{\sqrt{2S}} S_-\left(\vec{R}\right)\left|0\right\rangle$$

These $\left|\vec{R}\right\rangle$ states are not eigenstates of H, as

$$H\left|\vec{R}\right\rangle = E_o\left|\vec{R}\right\rangle + g\mu_B H\left|\vec{R}\right\rangle + S\sum_{R'} J\left(\vec{R}-\vec{R}'\right)\left(\left|\vec{R}\right\rangle - \left|\vec{R}'\right\rangle\right)$$

with $E_o = -NHg\mu_B S - \frac{1}{2}NZJS^2$ Z is the number of nearest neighbors.

This result if found using

$$S_x(\vec{R})S_x(\vec{R}) + S_y(\vec{R})S_y(\vec{R}) = \frac{1}{2}\left(S_+(\vec{R})S_-(\vec{R}) + S_-(\vec{R})S_+(\vec{R})\right)$$

$$S_z\left(\vec{R}'\right)\left|\vec{R}\right\rangle = S\left|\vec{R}\right\rangle \quad \vec{R}' \neq \vec{R}$$

$$S_z\left(\vec{R}\right)\left|\vec{R}\right\rangle = \left(S-1\right)\left|\vec{R}\right\rangle \quad \vec{R}' \neq \vec{R}$$

$$S_+\left(\vec{R}'\right)\left|\vec{R}\right\rangle = 0 \quad \vec{R}' \neq \vec{R}$$

$$S_-\left(\vec{R}'\right)S_+\left(\vec{R}\right)\left|\vec{R}\right\rangle = 2S\left|\vec{R}'\right\rangle$$

The hamiltonian H has the translationnal invariance of the lattice of the ferromagnet. As in the more familiar case of phonons, the following linear combinations of $\left|\vec{R}\right\rangle$ states are eigenstates of H:

$$\left|\vec{k}\right\rangle = \frac{1}{\sqrt{N}}\sum_{R} e^{i\vec{k}\vec{R}}\left|\vec{R}\right\rangle$$

where $\vec{k}$ are vetors of the reciprocal lattice. Indeed we can find that:

$$H\left|\vec{k}\right\rangle = \left(E_o + \hbar\omega\left(\vec{k}\right)\right)\left|\vec{k}\right\rangle$$

with

$$\hbar\omega\left(\vec{k}\right) = Hg\mu_B + JS\sum_{\delta}\left(1 - \cos\vec{k}\vec{\delta}\right)$$

where $\vec{\delta}$ connects one typical spin to its nearest neighbors.

The order of magnitude of $\hbar\omega\left(\vec{k}\right)$ is:

$$\hbar\omega\left(\vec{k}\right) \approx Hg\mu_B + JSa^2k^2$$

where $a = \left|\vec{\delta}\right|$

The spin wave $\left|\vec{k}\right\rangle$ states have a simple physical interpretation. Since the $\left|\vec{k}\right\rangle$ states are superpositions of $\left|\vec{R}\right\rangle$ states which are fully polarized except for one spin which is reduced by one, the $\left|\vec{k}\right\rangle$ states also have a magnetization NS-1. Furthermore, the transverse spin correlation function defined by:

$$\vec{S}_\perp(\vec{R})\vec{S}_\perp(\vec{R}') = \left(S_x(\vec{R})S_x(\vec{R}') + S_x(\vec{R})S_y(\vec{R}')\right)$$

has a simple expectation value in the $\left|\vec{k}\right\rangle$ states, namely :

$$\left\langle\vec{k}\left|\vec{S}_\perp(\vec{R})\vec{S}_\perp(\vec{R}')\right|\vec{k}\right\rangle = \frac{S}{N}\cos\left(\vec{k}\left(\vec{R} - \vec{R}'\right)\right) \quad \vec{R} \neq \vec{R}'$$

Hence the transverse component of each spin, $\vec{S}_\perp$ has a non-vanishing expectation value. The expectation value of the angle between two such transverse spin components is given by $\vec{k}\left(\vec{R} - \vec{R}'\right)$. The physical picture of these spin-wave states $\left|\vec{k}\right\rangle$ is consequently of the kind shown below :

Spin-wave resonances in metals

This section is presented as it reports experiments that are, to my mind, the most direct experimental evidence of spin-waves. Unlike neutron scattering, which requires an understanding of the notion of cross-section, of diffuse and coherent scattering, the microwave detection of spin-waves alluded to here, can be thought of simply as the detection of the resonant absorption of the microwave energy in a cavity.

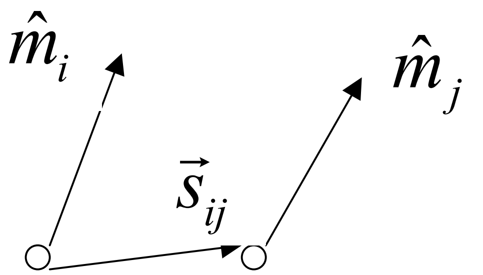
In the following, we want to refine the description of excitations in a ferromagnet by adding a term to the evolution of the magnetization which takes into account the effect of exchange. This term must be proportional to the distortion of the magnetization. Aharoni ("Introduction to the theory of Ferromagnetism", Amikam Aharoni, Int. Series of monographs on physics vol. 93, Oxford Science Publications, 1996) shows the exchange anisotropy field can be written as

$$\frac{C}{M_s}\nabla^2\vec{M}$$

The derivation goes along the following lines. Consider a Heisenberg exchange Hamiltonian:

$$E_{ex} = -\sum_{i,j} J_{ij}\vec{S}_i\vec{S}_j = -JS\sum_{neighbours}\cos\phi_{ij}$$

The angles are defined with the unit vector $\hat{m}$ on each site and $\hat{m}$ is along the magnetization $\vec{M}$.



In going to the continuous limit, it is assumed that the angles ϕ_{ij} are small. Then:

$$\left|\phi_{ij}\right| = \left|\hat{m}_i - \hat{m}_j\right| = \left|\left(\vec{s}_{ij}\vec{\nabla}\right)\hat{m}_i\right|$$

Finally, a simple cubic lattice is assumed, with six neighbours a distance “a” away from site i. The expansion of the cosine leaves a constant self energy term and a variation of energy due to the distortion of the magnetization field. This term is the exchange energy one needs to consider. The energy density for this exchange energy is then:

$$\frac{1}{2}\frac{2JS^2}{a}c\left(\left(\vec{\nabla}m_x\right)^2 + \left(\vec{\nabla}m_y\right)^2 + \left(\vec{\nabla}m_z\right)^2\right)$$

with c=1,2 and 4 for simple, bcc and fcc cubic structure respectively.

Deriving this term with respect to $\vec{M}$ yields the anisotropy field.

Our equation of evolution for the magnetization becomes :

$$\frac{\partial\vec{M}}{\partial t} = \gamma\vec{M}\times\vec{H} + D\vec{M}\times\nabla^2\vec{M}$$

The correspondence to an atomic model of exchange with a spacing a among sites and an exchange constant J is given by

$$D = 2Ja^2 / \hbar$$

We now assume that we have an infinite slab with the direction of the magnetic anisotropy axis and of the applied field normal to the slab. We set the z axis on the normal to the slab and consider small deviations from equilibrium, as above for the magnetostatic modes. We expect eigenmodes of the form:

$$\begin{pmatrix} m_x e^{i\omega t} e^{\pm ikz} \\ m_y e^{i\omega t} e^{\pm ikz} \\ M_0 \end{pmatrix}$$

Substituting in the equation of motion, keeping linear terms only, with the field taken to be:

$$\begin{pmatrix} 0 \\ 0 \\ H_i \end{pmatrix}$$

yields the eigenvalue equation :

$$\omega^2 = \left(\gamma H_i + Dk^2 M_0\right)^2$$

In the early days of ferromagnetic resonance, it was expected that strong field gradients would be necessary to excite these modes. In 1958, Kittel pointed out that the spins at the surface ought to be pinned by the surface magnetic anisotropy. (C. Kittel, "Excitation of Spin Wave resonance in a Ferromagnet by a uniform rf field", Phys. Rev. 110(6), 1295(1958)) In order to examine the extent to which these modes can be excited with a uniform rf field, we consider the field to be :

$$\begin{pmatrix} h_x \\ h_y \\ H_i \end{pmatrix}$$

so that the equation of motion is :

$$\begin{pmatrix} \dfrac{\partial m_x}{\partial t} \\ \dfrac{\partial m_y}{\partial t} \\ \dfrac{\partial m_z}{\partial t} \end{pmatrix} = \begin{vmatrix} \hat{x} & \gamma m_x & h_x \\ \hat{y} & \gamma m_y & h_y \\ \hat{z} & \gamma m_z & H_i \end{vmatrix} + D \begin{vmatrix} \hat{x} & m_x & \nabla^2 m_x \\ \hat{y} & m_y & \nabla^2 m_y \\ \hat{z} & m_z & 0 \end{vmatrix}$$

As usual, we consider

$$h_{\pm} = h_{\pm} + ih_{\pm}$$

and

$$m_{\pm} = m_{\pm} + im_{\pm}$$

We have

$$\frac{\partial m_+}{\partial t} = iM_0 h_+ - iH_i m_+ + iDM_0 \nabla^2 m_{\pm}$$

Assume a solution for m_+ of the form

$$m_+ = e^{i\omega t} \sum_p a_p \sin(k_p z)$$

If one side of the film is at z=0 and the other at z=L, then the pinning of the spins at the surface impose :

$$k_p = \frac{p\pi}{L}$$

The rf field is supposed homogeneous:

$$h_+ e^{i\omega t}$$

So the amplitudes a_p must satisfy:

$$\sum_p a_p (\omega + \omega_p) \sin(k_p z) = \gamma M_0 h_+$$

Multiplying by $\sin(k_m z)$ and integrating from 0 to L gives:

$$a_p (\omega + \omega_p) \frac{L}{2} = \frac{\gamma M_0 h_+}{\pi m / L} (1 - \cos(m\pi))$$

The modes with even m are not excited. The oscillator strength of the odd modes is given by

$$a_p = \frac{4\gamma M_0}{(\omega + \omega_p)\pi m} h_+$$

The larger m, the weaker the resonance. This is intuitive: the higher m, the smaller the wave length of the mode, the harder it is to excite it with a uniform field!

In a typical microwave absorption experiment, the frequency ω is kept constant, for practical reasons. The field H_i is swept instead.

Soon after the suggestion of Kittel that spin waves could be excited in a thin film with a homogeneous field, the spin wave resonance of a permalloy thin film was detected (Figure below). (M.H. Seavey, P.E. Tannenwald, "Direct Observation of Spin-Wave Resonance", Phys. Rev. Lett, 1(5), 168 (1958))

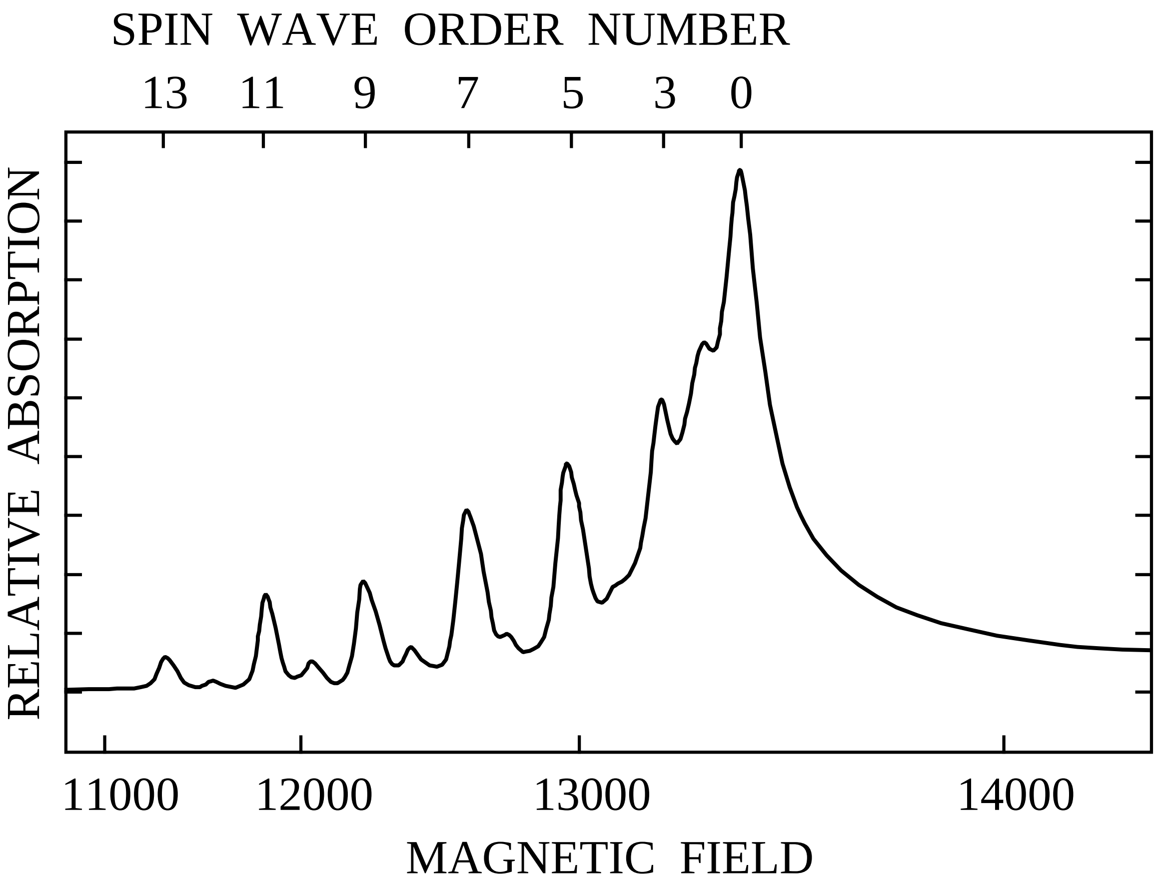
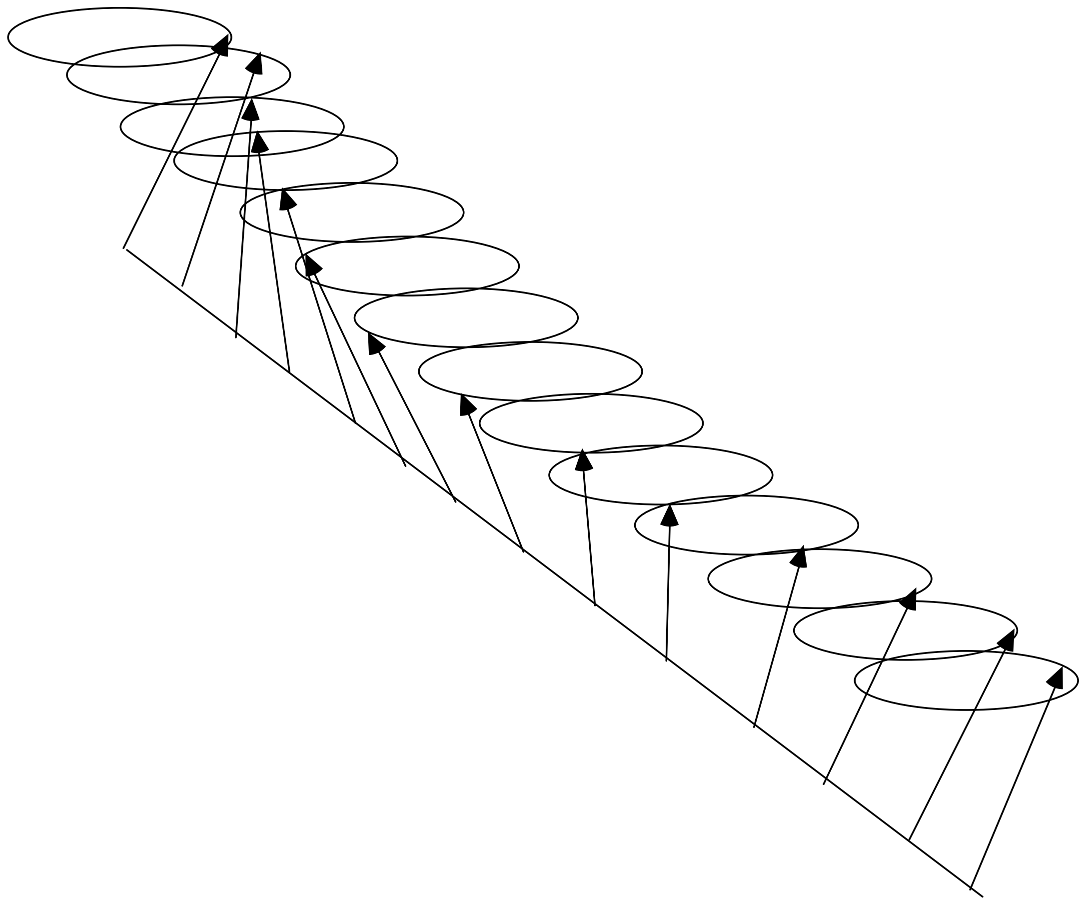


Figure : spin wave resonance in permalloy

It is clear that with such a highly structured spectrum, a direct estimate of the exchange constant can be made quite nicely, provided M_o is known.

It may be useful to keep some order of magnitudes in sight: for the experiment above, the film was about half a micrometer thick, and the spacing between two adjacent lines of the spectrum was of the order of 500 Oe. The mode with m=5 has then a wave length of about 100 nm.

These modes do depend on the exchange constant, and on the size of the sample. The magnetostatic modes depend on the shape of the sample, but not on its size! The magnetostatic modes are shown to be the proper long-wave length extension of the spin wave modes, as they should.



Holstein-Primakov description of spin-waves

This section is a brief account of the chapter 5.1 of Mattis' « Quantum Theory of Magnetism », who himself made an account of the review of spin waves of Van Vleck and Van Kranendonk in Rev. of Mod. Phys. vol.30, 1, (1958)

We use again an exchange hamiltonian to describe a ferromagnet.

$$H = -J \sum_{\substack{i>j= \\ \text{nearest neighb.}}}\sum \mathbf{S}_i \bullet \mathbf{S}_j - Hg\,\mu_B \sum_i S_j^z \qquad J>0$$

The exchange interaction here is isotropic, with

$$\mathbf{S}_i \bullet \mathbf{S}_j \equiv S_i^x S_j^x + S_i^y S_j^y + S_i^z S_j^z = \frac{1}{2}\left(S_i^+ S_j^- + S_j^+ S_i^-\right) + S_i^z S_j^z .$$

In addition to this exchange interaction, we want to take into account long-range interactions, because we are interested in the long-wave length limit of spin waves. The lowest order long-range interactions are dipolar in nature, given by:

$$\frac{1}{2}\sum_i\sum_j D_{ij}\,\frac{\mathbf{S}_i \bullet \mathbf{S}_j \mathbf{r}_{ij}^2 - 3\;\mathbf{S}_i \bullet \mathbf{r}_{ij}\mathbf{S}_j \bullet \mathbf{r}_{ij}}{\mathbf{r}_{ij}^2}$$

where

$$D_{ij} = g^2\mu_B^2 r_{ij}^{-3}$$

for the magnetic dipole-dipole interation. Other couplings may have the same form, such as the indirect coupling associated with the spin-orbit coupling.

We are now setting out to use the so-called Holstein-Primakov representation of spins operators in order to describe spin-waves as harmonic oscillators. We designate the eigenstate of $S_{z,i}$ by

$$|n_i\rangle, \quad n = 0,...,2s+1$$

. We can think of n_i as numbering the eigenstates of starting S_z from —s, then —s+1, etc… until +s, that is, there are 2s+1 such states. The spin operators have the following matrix elements in this basis:

$$< n_i \ |S_j^x| \ n_i + 1> = < n_i + 1 \ |S_i^x| \ n_i >^* = \frac{1}{2}\sqrt{(n_i+1)(2s-n_i)}$$

$$< n_i | S_i^y | n_i + 1 > = < n_i + 1 | S_i^y | n_i >^* = -\frac{1}{2} i \sqrt{(n_i+1)(2s-n_i)}$$

$$< n_i | S_i^z | n_i > = s - n_i, \quad 0 \le n_i \le 2s, \quad \hbar = 1.$$

Consider then the eigenstates $|n_i\rangle$ of the hamiltonian of a harmonic oscillator:

$$\frac{p_i^2}{2m} + \frac{1}{2}m\omega^2 x_i^2 - \frac{1}{2}\hbar\omega | n_i > = n_i\hbar\omega | n_i > .$$

The matrix elements of the positon and momentum operators are:

$$< n_i | x_i | n_i + 1 > = < n_i + 1 | x_i | n >^* = \sqrt{\frac{\hbar}{2m\omega}(n_i+1)}$$

$$< n_i | p_i | n_i + 1 > = < n_i + 1 | p_i | n_i >^* = -\sqrt{\frac{\hbar m\omega}{2}(n_i+1)}$$

Clearly, these matrix elements will approach those of the angular momentum if we rescale the position and momentum operators according to:

$$Q_i = x_i\sqrt{\frac{m\omega}{\hbar}} \quad P_i = p_i\sqrt{\frac{1}{\hbar m\omega}} \quad \left[P_i,\ Q_j\right] = \delta_{ij}\frac{1}{i}$$

Indeed we then have:

$$< n_i | Q_i\sqrt{s} | n_i + 1 > = < n_i + 1 | Q_i\sqrt{s} | n_i >^* = \frac{1}{2}\sqrt{(n_i+1)2s}$$

$$< n_i | P_i\sqrt{s} | n_i + 1 > = < n_i + 1 | P_i\sqrt{s} | n_i >^* = -\frac{1}{2} i\sqrt{(n_i+1)2s}$$

$$< n_i | s - \frac{1}{2}(P_i^2 + Q_i^2 - 1) | n_i > = s - n_i$$

We are going to limit ourselves from now on to small values of n_i. In view of the theory of spin waves given above for the case without long range interactions, taking small values n_I

corresponds to small deviations at any spin site from the ground state value $n_i=0$. Consequently to the extent that one limits n_i to small values, one as

$$S_i^x = Q_i\sqrt{s}, \qquad S_i^y = P_i\sqrt{s}, \qquad S_i^z = s - \frac{1}{2}(P_i^2 + Q_i^2 - 1)$$

We proceed with expressing the exchange hamiltonian in this representation, known as the Holstein-Primakov representation of angular momentum. We substitute in the hamiltonian these expressions for the angular momentum. As we want to look at the lowest order effect, we neglected in this substitution, terms that are cubic and quartic in these operators. Hence the exchange hamiltonian only becomes :

$$H_{lin} = E_0 + g\mu_B(H+H_0)\sum_i \frac{1}{2}(P_i^2 + Q_i^2 - 1) - Js\sum_{\substack{\text{nearest}\\ \text{neighb.}}}(P_iP_j + Q_iQ_j)$$

where

$$H_0 = \frac{Jsz}{g\mu_B} \quad , \qquad E_0 = -\ NHg\mu_B s - \frac{1}{2}\ NzJs^2$$

This is the molecular field of Weiss' effective field approximation. E_0 is the energy of the system when all the spins are aligned in the applied field.

We have thus obtained a Hamiltonian, which has some resemblance to that of a harmonic oscillator, except for the last terms. Nonetheless, we make the expansion in plane waves in analogy to what we did in the previous section:

$$Q_i = \frac{1}{\sqrt{N}}\sum_k e^{i\mathbf{k}\bullet\mathbf{R}_i}Q_k \qquad P_i = \frac{1}{\sqrt{N}}\sum_k e^{i\mathbf{k}\bullet\mathbf{R}_i}P_k$$

Note that:

$$Q_k^* = Q_{-k} \ \text{ etc.} \qquad \left[P_k^*,\ Q_{k'}\right] = \frac{\delta_{kk'}}{i}$$

where the **k**'s belong to the reciprocal lattice. This expansion produces an exchange hamiltonian of the form:

$$H_{lin} = \sum_k \frac{1}{2}(P_k^*P_k + Q_k^*Q_k - 1)\hbar\omega(\mathbf{k}) + E_0 = \sum_k n_k\hbar\omega(\mathbf{k}) + E_0 .$$

where

$$n_k \equiv \frac{1}{2}(P_k^*P_k + Q_k^*Q_k - 1)$$

$$\hbar\omega(\mathbf{k}) = Hg\mu_B + Js\sum_\delta (1-\cos\ \mathbf{k}\bullet\delta) \cong\ Hg\mu_B + Jsa^2\mathbf{k}^2 + 0(\mathbf{k}^4)$$

It can be verified that the new operators $Q_i = \frac{1}{\sqrt{N}}\sum_k e^{i\mathbf{k}\bullet\mathbf{R}_i}Q_k \quad P_i = \frac{1}{\sqrt{N}}\sum_k e^{i\mathbf{k}\bullet\mathbf{R}_i}P_k$

have the commutation properties of position and momentum operators, and that operators of different **k**'s commute. So we have a sum of uncoupled harmonic oscillators.

We now address the issue of the effect of the long range couplings on the spin-wave spectrum. The direction cosines of a spin i coupled to a spin j are given by :

$$\frac{\mathbf{r}_{ij}}{r_{ij}} = (\alpha_{ij}, \beta_{ii}, \gamma_{ij})$$

The dipolar hamiltonian can be decomposed in three parts:

$$H_{d,0} = \frac{1}{2} s^2 \sum_i \sum_j D_{ij}(1 - 3\gamma_{ij}^2)$$

$$H_{d,1} = -\ 3s^{3/2} \sum_i \sum_j D_{ij}(\alpha_{ij}\gamma_{ij}Q_j + \beta_{ij}\gamma_{ij}P_j)$$

$$H_{d,2} = \frac{1}{2} s \sum_i \sum_j D_{ij}\left[(1-3\alpha_{ij}^2)Q_iQ_j - 3\alpha_{ij}\beta_{ij}(Q_iP_j + P_iQ_j) + (1-3\beta_{ij}^2)P_iP_j - (1-3\gamma_{ij}^2)(P_i^2 + Q_i^2 - 1)\right]$$

The first term is associated with the demagnetizing field. For an ellipsoid with a field in the z direction, the demagnetizing coefficient is given by:

$$N_z = a^3 \sum_j \frac{1}{r_{ij}^3}(1 - 3\gamma_i^2) + \frac{4\pi}{3}$$

The magnetization of the ellipsoid is uniform, given by:

$$M_0 = \frac{g\mu_B s}{a^3}$$

Hence:

$$H_{d,0} = -\ \frac{1}{2}\ VM_0\left(\frac{4\pi}{3}M_0 - N_z M_0\right).$$

The second term cancels for any point of symmetry, such as in a cubic crystal. It would be small anyway, as shown by Mattis.

We are left with $H_{lin} = \sum_k \frac{1}{2}(P_k^* P_k + Q_k^* Q_k - 1)\hbar\omega(\mathbf{k}) + E_0 = \sum_k n_k \hbar\omega(\mathbf{k}) + E_0$. and

$$H_{d,2} = \frac{1}{2} s \sum_i \sum_j D_{ij}\left[(1-3\alpha_{ij}^2)Q_iQ_j - 3\alpha_{ij}\beta_{ij}(Q_iP_j + P_iQ_j) + (1-3\beta_{ij}^2)P_iP_j - (1-3\gamma_{ij}^2)(P_i^2 + Q_i^2 - 1)\right]$$

. Again we operate the transformation to plane waves. Substitution into H_{lin} and $H_{d,2}$ produces :

$$H \to \frac{1}{2} \sum_{\mathbf{k}} \left[A(\mathbf{k})Q_{\mathbf{k}}^* Q_{\mathbf{k}} + B(\mathbf{k})P_{\mathbf{k}}^* P_{\mathbf{k}} + 2C(\mathbf{k})Q_{\mathbf{k}}^* P_{\mathbf{k}}\right] + \text{const}$$

where

$$A(\mathbf{k}) = \hbar\omega(\mathbf{k}) + A_{xx}(\mathbf{k}) - A_{zz}(\mathbf{0})$$

$$B(\mathbf{k}) = \hbar\omega(\mathbf{k}) + A_{yy}(\mathbf{k}) - A_{zz}(\mathbf{0})$$

$$C(\mathbf{k}) = A_{xy}(\mathbf{k}).$$

$A_{xx}(\mathbf{k}) = \frac{S}{N}\sum_{i,j} D_{ij}(1 - 3\alpha_{ij}^2)\ e^{i\mathbf{k}\bullet\mathbf{R}_{ij}}$, likewise for $A_{xx}(\mathbf{k}) = \frac{S}{N}\sum_{i,j} D_{ij}(1 - 3\alpha_{ij}^2)\ e^{i\mathbf{k}\bullet\mathbf{R}_{ij}}$

$A_{xx}(\mathbf{k}) \sim \left(1 - 3\frac{k_x^2}{k^2}\right)\left[1 - \frac{3j_1(kR)}{kR}\right]$, likewise for $A_{xx}(\mathbf{k}) = \frac{S}{N}\sum_{i,j} D_{ij}(1 - 3\alpha_{ij}^2)\ e^{i\mathbf{k}\bullet\mathbf{R}_{ij}}$

and $\hbar\omega(\mathbf{k})$ is defined as above. Size effects are to be expected in these dipolar sums. See Cohen and Keffer, Phys. Rev. 99, 1128, 1135(1955). For example, the case of a sphere gives for a position $\mathbf{r_i}$ near the center of the crystal :

$$A_{xx}(\mathbf{k}) \sim \left(1 - 3\frac{k_x^2}{k^2}\right)\left[1 - \frac{3j_1(kR)}{kR}\right]$$

We can operate once again a canonical transformation to transform the hamiltonian into that of a sum of harmonic oscillator. The canonical transform is of the form:

$$P'_{\mathbf{k}} = a_{\mathbf{k}}P_{\mathbf{k}} + b_{\mathbf{k}}Q_{\mathbf{k}} \qquad Q'_{\mathbf{k}} = c_{\mathbf{k}}Q_{\mathbf{k}} + d_{\mathbf{k}}P_{\mathbf{k}}$$

$$[P'^{*}_{\mathbf{k}_1},\ Q'_{\mathbf{k}_2}] = \frac{\delta_{\mathbf{k}_1\mathbf{k}_2}}{i}$$

If the dipolar sums in $A(\mathbf{k}) = \hbar\omega(\mathbf{k}) + A_{xx}(\mathbf{k}) - A_{zz}(\mathbf{0})$ are known, the coefficients defining this canonical transform can be determined so that the hamiltonian becomes:

$$H = \sum \frac{1}{2}\ (P'^{*}_{\mathbf{k}}P'_{\mathbf{k}} + Q'^{*}_{\mathbf{k}}Q'_{\mathbf{k}} - 1)\hbar\omega'(\mathbf{k}) + \text{const}$$

where the spin-wave spectrum now becomes :

$$\hbar\omega'(\mathbf{k}) = \sqrt{A(\mathbf{k})B(\mathbf{k}) - C^2(\mathbf{k})}$$

In a ferromagnetic resonance experiment, the rf field is fairly homogeneous and it can excite only the spin waves modes near $\mathbf{k} = 0$. The resonance frequency is given by $\hbar\omega'(\mathbf{k}) = \sqrt{A(\mathbf{k})B(\mathbf{k}) - C^2(\mathbf{k})}$ in the limit when k tends to zero. In this limit, the $A_{xx}(\mathbf{k})$ terms can be neglected, for the same reasons that $H_{d,1}$ was. The other sums can be identified with demagnetizing factors and the final result is:

$$\hbar\omega'(\mathbf{0}) = g\mu_B\sqrt{[H + (N_x - N_z)M_0]\ [H + (N_y - N_z)M_0]}.$$

This is a fully quantum mechanical derivation of the semi-classical result discussed in the introduction.